# Engineering Notes

*ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure or vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).*

## Calculation of Lift and Induced Drag from Sparse Span Loading Data

J. L. Lundry*
*Boeing Commercial Airplane Company, Seattle, Wash.*

### Nomenclature

$A_i$ = Glauert sine series coefficients
$AR$ = aspect ratio
$C_i$ = constraint condition
$C_{D_i}$ = induced drag coefficient
$C_L$ = lift coefficient
$E$ = nondimensionalized strain energy $[=\pi^3 e/24\,\ell(\pi/2)]$
$L$ = given span loading
$M$ = number of given span loading points
$N$ = number of series coefficients
$a$ = coefficients of minimum strain energy polynomial for span loading
$b$ = wing span
$c$ = local wing chord
$c_\ell$ = wing section lift coefficient
$e$ = strain energy in the theta-plane
$k$ = induced drag efficiency parameter $(=C_L^2/\pi AR\, C_{D_i})$
$\ell$ = span loading $(=cc_\ell/4b)$
$y$ = spanwise coordinate
$\theta$ = transformed spanwise coordinate (= arccos $\eta$)
$\eta$ = nondimensional span station $(=y/b/2)$
$\underline{\eta}$ = spanwise centroid of lift
$\lambda$ = Lagrange multiplier

### Introduction

AIRPLANE designers frequently have only a limited quantity of loads data available from wind tunnel or flight tests to define span loading. For these situations and for analytical methods restricted to a modest number of span stations, it would be useful to be able to calculate induced drag when span loading is defined at only a few discrete span stations. The key to the problem is the identification of a suitable curve-fitting algorithm to fit the given data. This Note presents a strain energy method suitable for use on digital computers.

### Analysis

Glauert's sine series for span loading[1] can be written

$$\ell(\theta)=\sum_{i=1}^{N} A_i \sin i\theta \tag{1}$$

Table 1 gives expressions for gross characteristics of the loading which can be obtained from the series coefficients $A_i$. These include the strain energy in the theta-plane which is proportional to

$$e=\int\left(\frac{d^2\ell}{d\theta^2}\right)^2 d\theta \tag{2}$$

Received Aug. 10, 1976; revision received Sept. 27, 1976.
Index category: Aircraft Aerodynamics (including Component Aerodynamics).

*Senior Specialist Engineer, Aerodynamics Research Unit. Member AIAA.

where the limits of integration include the full span for unsymmetric loading and are restricted to the semispan for symmetric span loading.

In this Note, the coefficients are determined by minimizing the strain energy while constraining $\ell(\theta)$ to pass through the given loading points, $L(\theta_j)$, $j=1, M$. The theory of Lagrange multipliers gives a necessary condition which reduces to

$$\pi i^4 A_i + \sum_{j=1}^{M} \lambda_j \sin i\theta_j = 0 \quad j=1, N \tag{3}$$

for unsymmetric span loading; the loading constraints are

$$\sum_{i=1}^{N} A_i \sin i\theta_j = L(\theta_j) \quad j=1, M \tag{4}$$

For symmetric span loading, Eqs. (3) and (4) change to

$$\frac{\pi}{2}(2i-1)^4 A_{2i-1} + \sum_{j=1}^{M} \lambda_j \sin(2i-1)\theta_j = 0 \quad i=1, N \tag{5}$$

and

$$\sum_{i=1}^{N} A_{2i-1} \sin(2i-1)\theta_j = L(\theta_j) \quad j=1, M \tag{6}$$

In both cases, these equations are linear and complete in $A$ and $\lambda$.

For symmetric span loading, a limiting case of this algorithm has been found. This case provides a means of normalizing $e$, and defines the shape of the limiting span loading. Instead of assuming a functional relationship between $\ell$ and $\theta$, [Eq. (1)], assume that the loading at the wing tip is zero,

$$\ell(0)=0 \tag{7}$$

that the loading at the root chord is known,

$$\ell(\pi/2)=\text{constant} \tag{8}$$

and that the slope of the loading at the root chord is zero,

$$\frac{d\ell}{d\theta}(\pi/2)=0 \tag{9}$$

We wish to find the loading $\ell(\theta)$ for which $e$ is minimum and for which Eqs. (7-9) are satisfied. This constitutes a variational calculus problem, for which analysis shows that

$$\frac{d^4\ell}{d\theta^4}=0 \tag{10}$$

and

$$\ell(\theta)=\sum_{i=0}^{3} a_i\theta^i \tag{11}$$

Equation (7) shows that $a_0$ is zero, but we are left with three unknown coefficients in Eq. (11), and only two conditions.

At this point, we recast the problem as an ordinary minimization of a function of the three undetermined

Table 1 Span loading characteristics

| Parameter | Symmetric span loading | Unsymmetric span loading |
|---|---|---|
| $\ell(\theta) \equiv \dfrac{cc_\ell}{4b}$ | $\sum_{i=1}^{N} A_{2i-1} \sin(2i-1)\theta$ | $\sum_{i=1}^{N} A_i \sin i\theta$ |
| $C_L$ | $\pi AR\, A_1$ | $\pi\, AR\, A_1$ |
| $C_{D_i}$ | $\pi\, AR \sum_{i=1}^{N} (2i-1)\, A_{2i-1}^2$ | $\pi AR \sum_{i=1}^{N} i\, A_i^2$ |
| $\bar{\eta}$ | $\dfrac{4}{\pi A_1} \sum_{i=1}^{N} \dfrac{(-1)^i A_{2i-1}}{(2i-3)(2i+1)}$ (semispan) | $\dfrac{A_2}{2\, A_1}$ (full span) |
| $E$ | $\dfrac{\pi}{4} \sum_{i=1}^{N} (2i-1)^4 A_{2i-1}^2$ | $\dfrac{\pi}{2} \sum_{i=1}^{N} i^4 A_i^2$ |

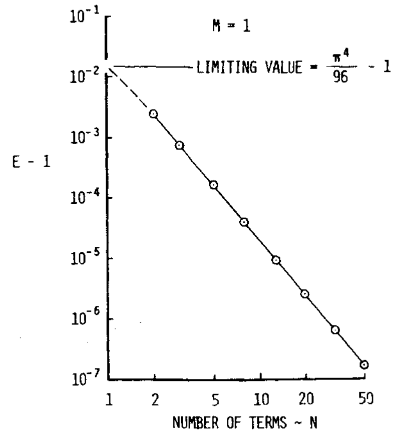


**Fig. 1 Error in strain energy as a function of the number of series coefficients.**

variables $a_i$, the variational calculus having shown us the necessary functional loading relationship [Eq. (11)]. This produces

$$a_1 = 3/\pi\ \ell(\pi/2) \tag{12a}$$

$$a_2 = 0 \tag{12b}$$

$$a_3 = -4/\pi^3\ \ell(\pi/2) \tag{12c}$$

and

$$e_{\min} = 24/\pi^3\ \ell^2(\pi/2) \tag{13}$$

The latter quantity is used to normalize $e$. The span loading defined in Eqs. (12) is not shown, due to space restrictions; however, it is near elliptical, and

$$\frac{e_{\text{elliptical}}}{e_{\min}} = \pi^4/96 \doteq 1.015 \tag{14}$$

## Results

The influence of the number of sine-spline series terms $N$ and the number of points used to specify the loading $\ell$ is shown for simple test cases in Figs. 1 and 2. If a single point is used to specify the loading, then the nondimensional strain energy $E$ should approach 1 as $N$ is increased. This trend is shown in Fig. 1.

For $N=40$, Fig. 2 shows one effect of increasing the number of points $M$ used to specify an elliptical span loading. The induced drag efficiency parameter $k$ is 1.0 for elliptical spar loading, and Fig. 2 shows that this limit is approached whether the loading specification points are spaced equally ir $\theta$ or in $\eta$.

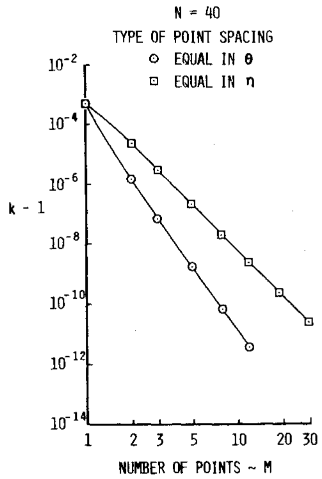


**Fig. 2 Error in efficiency parameter as a function of the number of loading definition points: elliptical span loading.**

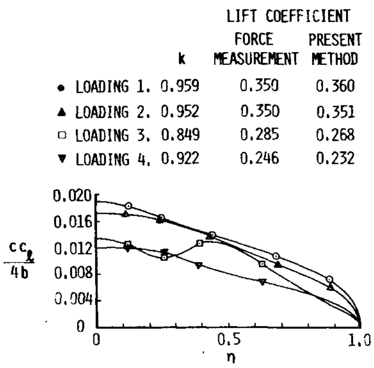


**Fig. 3 Examples of span loading fairing.**

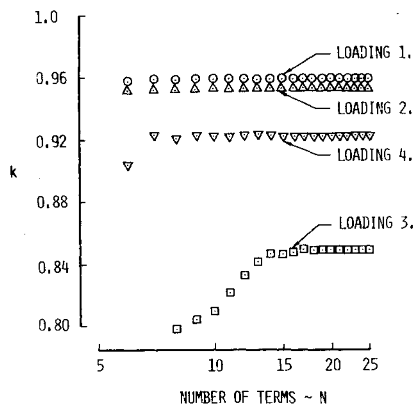


**Fig. 4 Variation of induced drag efficiency with number of series coefficients.**

The sine-spline method has been applied to four spar loadings obtained from wind tunnel pressure measurements Figure 3 shows the measured loading data and the curve: faired by the method; the ordinates of each curve and th corresponding data have been multiplied by a factor in orde that all four curves might be shown in one plot. The sine spline fairing is reasonable for each of the four loadings although one might be tempted to raise the faired curve o loading 3 somewhat between the wing tip and the most out board datum. For each fairing, the lift coefficient i reasonably close to the experimental lift coefficient obtaine from force data; the largest discrepancy occurs for loading 3

The trend of $k$ with increasing $N$ is shown in Fig. 4 for each of these four loadings.

The sine-spline induced drag method has been in production use at Boeing for some time, and has given satisfactory results in all cases of which the author is aware. The method has proved useful in separating induced drag from lift-dependent drag measured by wind tunnel tests. It has also been used to calculate the induced drag of span loadings determined by theory. The method is coded in FORTRAN for the CDC 6600 digital computer, requires about 30,000 (octal) core locations, and requires about one second of CP time per case.

### Acknowledgment

This work has been accomplished under the sponsorship of the Independent Research and Development program of the Boeing Commercial Airplane Company. The author acknowledges the contribution of S. J. Cowan, who provided the variational analysis leading to Eq. (10).

### Reference

[1] Glauert, H., Chap. XI, *The Elements of Aerofoil and Aeroscrew Theory*, second edition, Cambridge University Press, London, 1948.

# Airspeed Stability under Wind Shear Conditions

Malcolm J. Abzug*
*Aeronautical Consultant Associates, Pacific Palisades, Calif.*

### Nomenclature

$C_D$ = drag coefficient
$h$ = height perturbation
$m$ = mass
$\rho$ = density
$s$ = Laplace transform variable
$S$ = wing area
$t$ = time
$T$ = thrust
$\tau$ = airspeed time constant
$u, w$ = airspeed and vertical velocity perturbations
$U_0$ = equilibrium airspeed
$X, Z$ = aerodynamic forces along $X$ and $Z$ axes
$X_u^*$ = stability derivative $(\partial X/\partial u)/m$, with thrust
$X_w$ = stability derivative $(\partial X/\partial w)/m$
$Z_u^*$ = stability derivative $(\partial Z/\partial u)/m$, with thrust
$Z_w$ = stability derivative $(\partial Z/\partial w)/m$

### Introduction

A LANDING aircraft may encounter wind shear or a variation with altitude (and time) of the horizontal wind components along and normal to the landing approach path. The effect of the component along the path on altitude control has received recent attention because of aircraft accidents in which wind shear is a suspected cause. Investigations of wind shear effects usually have relied on large-scale computer simulations[1] or on the use of flight simulators. It is the purpose of this Note to point out that some insight into an aspect of the problem is furnished by a simple analytic model.

Received Nov. 18, 1976.
Index categories: Aircraft Gust Loading and Wind Shear; Aircraft Handling, Stability, and Control; Aircraft Landing Dynamics.
*Consultant. Fellow AIAA.

### Airspeed Stability

Horizontal wind strength component variations along an aircraft's landing path can create sudden altitude loss when the change is equivalent to a reducing head wind. Aircraft inertia will cause a sudden loss in head wind to result in the same loss in airspeed. Both lift and drag will decrease. The lift reduction will cause the aircraft to settle below the glide path, whereas the drag reduction will cause the aircraft to accelerate toward the original airspeed. The more rapid the airspeed recovery the less loss in altitude will result. Although the pilot can increase the acceleration toward the original airspeed by increasing thrust, it is pertinent to examine the aircraft's inherent tendency to regain airspeed, or its airspeed stability under wind shear conditions.

### Mathematical Model

Airspeed stability in the sense described is found by a solution of the homogeneous linear differential equations of motion.[2] Pitch attitude perturbations are neglected on the assumption that the pilot or automatic pilot holds nearly constant attitude during the disturbance. Longitudinal and vertical velocity responses to an initial airspeed perturbation $u(0+)$ are, in Laplace transforms

$$u(s)/u(0+) = (s - Z_w)/\Delta \tag{1}$$

$$w(s)/u(0+) = Z_u^*/\Delta \tag{2}$$

where

$$\Delta = (s - X_u^*)(s - Z_w) - X_w Z_u^* \tag{3}$$

Conditions at the start of the motion described by Eqs. (1) and (2) correspond to the aircraft immediately after its airspeed has been perturbed by the amount $u(0+)$ from equilibrium, as by a sudden change in head wind.

The characteristic equation (3) factors into small and large real roots, for typical values of the stability derivatives. The small root may be called the "airspeed" root because it dominates the longitudinal motion governed by Eq. (1). The large real root contributes to the vertical or plunge motion. This root is equal approximately to $s - Z_w$. Time solutions of Eqs. (1) and (2) for a large jet transport in landing approach are shown in Fig. 1. The integral of the perturbed vertical velocity $w$ also is shown. This is equivalent to a height perturbation $h$ from the glide path. The airspeed perturbation $u$ is almost a pure exponential. Under the assumptions, the height loss $h$ approaches a fixed value of 4.53 ft/ft/sec of initial airspeed perturbation.

A single-degree-of-freedom representation of the disturbed longitudinal motion is suggested by the minor contribution to that motion of the large real root related to the vertical or plunge motion. The single-degree-of-freedom solution is

$$u(s)/u(0+) = 1/(s - X_u^*) \tag{4}$$

or

$$u(t)/u(0+) = e^{-t/\tau}$$

**Table 1 Airspeed time constants for two commerical aircraft**

| | Jet transport | Light twin |
|---|---|---|
| Airspeed, kt | 150.0 | 88.4 |
| Gross weight, lb | 245,400 | 3330 |
| Wing area, ft$^2$ | 3123 | 178 |
| $C_D$ | 0.167 | 0.090 |
| $\partial T/\partial u$, lb/fps | −20.24 | −0.74 |
| Time constant, $\tau$, sec | 22.8 | 16.1 |